

ON EXTREME POINTS OF MEASURES WHICH IMPLEMENT AN ISOMETRIC EMBEDDING OF MODEL SPACES

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ABSTRACT. In 1996 A. Aleksandrov solved an isometric embedding problem for model spaces K_Θ with an arbitrary inner function Θ . We find all extreme points of this convex set of measures in the case when Θ is a finite Blaschke product, and obtain some partial results for generic inner functions.

INTRODUCTION

In [2] A. Aleksandrov settled the isometric embedding problem for model spaces K_Θ . Precisely, given an arbitrary inner function Θ on the unit disk \mathbb{D} , one seeks for the collection of all finite, positive, Borel measures σ on the unit circle \mathbb{T} so that the identity operator (embedding) of the model space $K_\Theta := H^2 \ominus \Theta H^2$ to the space $L^2_\sigma(\mathbb{T})$ is isometric. In other words, the equality

$$\langle f, g \rangle_\sigma := \int_{\mathbb{T}} f(t) \overline{g(t)} \sigma(dt) = \int_{\mathbb{T}} f(t) \overline{g(t)} m(dt) = \langle f, g \rangle_m$$

holds for each $f, g \in K_\Theta$. Here m is the normalized Lebesgue measure on \mathbb{T} .

Denote this set of measures by $\Sigma(\Theta)$, and the unit ball of H^∞ (the Schur class) by \mathcal{S} . The result of Aleksandrov looks as follows.

Theorem A. $\sigma \in \Sigma(\Theta)$ if and only if there is a real number β and a Schur function $\omega \in \mathcal{S}$ so that

$$(0.1) \quad \frac{1 + \Theta(z)\omega(z)}{1 - \Theta(z)\omega(z)} = i\beta + \int_{\mathbb{T}} \frac{t + z}{t - z} \sigma(dt).$$

Relation (0.1) can be viewed as a counterpart of the Nevanlinna parametrization in the Hamburger moment problem, see [1, Theorem 3.2.2].

Remark 0.1. The function ω in (0.1) is an independent parameter, which runs over the whole class \mathcal{S} . Both β and σ in (0.1) are uniquely determined by ω ,

$$\beta = \frac{2 \operatorname{Im} (\omega(0)\Theta(0))}{|1 - \Theta(0)\omega(0)|^2}.$$

Conversely, if two triples $\{\omega_j, \beta_j, \sigma\}$, $j = 1, 2$, satisfy (0.1), then $\omega_1 = \omega_2$ and $\beta_1 = \beta_2$. For instance, $\sigma = m$ enters the only triple $\{0, 0, m\}$. So, equality

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(0.1) generates a bijection I

$$(0.2) \quad I : \Sigma(\Theta) \rightarrow \mathcal{S}, \quad I(\sigma) = \omega.$$

The set $\Sigma(\Theta)$ is easily seen to be a convex set which is compact in *-weak topology of the space $\mathcal{M}_+(\mathbb{T})$ of finite, positive, Borel measures on \mathbb{T} . The study of the set $\Sigma_{ext}(\Theta)$ of extreme points for $\Sigma(\Theta)$ seems quite natural. This is exactly the problem we address here. A point $\sigma \in \Sigma(\Theta)$ is said to be an *extreme point* of $\Sigma(\Theta)$ if

$$(0.3) \quad \sigma = \frac{\sigma_1 + \sigma_2}{2}, \quad \sigma_j \in \Sigma(\Theta) \Rightarrow \sigma_1 = \sigma_2 = \sigma.$$

Equivalently, there is no nontrivial representation of σ as a convex linear combination of two measures from $\Sigma(\Theta)$.

We say that a measure $\sigma \in \mathcal{M}_+(\mathbb{T})$ has a finite support if

$$\sigma = \sum_{j=1}^p s_j \delta(t_j), \quad s_j > 0, \quad \text{supp } \sigma = \{t_j\}_{j=1}^p, \quad t_j = t_j(\sigma),$$

and write $|\text{supp } \sigma| = p$ for such measures. Denote by $\Sigma_f(\Theta)$ the set of measures with the finite support in $\Sigma(\Theta)$. It is clear that $\Sigma_f(\Theta)$ is nonempty if and only if both $\Theta = B$ and ω are finite Blaschke products (FBP).

Here is our main result.

Theorem 0.2. *Let B be a FBP of order $n \geq 1$. The measure $\sigma \in \Sigma_{ext}(B)$ if and only if $\sigma \in \Sigma_f(B)$ and*

$$(0.4) \quad n \leq |\text{supp } \sigma| \leq 2n - 1.$$

Denote by $\mathcal{S}_{ext}(\Theta)$ the set of all $\omega \in \mathcal{S}$ so that the corresponding σ in (0.1) belongs to $\Sigma_{ext}(\Theta)$. Equivalently, $\mathcal{S}_{ext}(\Theta)$ is the image of $\Sigma_{ext}(\Theta)$ under transformation I (0.2). The above result can be paraphrased as follows.

Theorem 0.3. *Let B be a FBP of order $n \geq 1$. The set $\mathcal{S}_{ext}(B)$ agrees with the set of all FBP's of the order at most $n - 1$.*

The case of generic inner functions Θ is much more delicate. We can supplement the above result with the following

Theorem 0.4. *Let Θ be an inner functions with n distinct zeros. Then each FBP of order at most $n - 1$ belongs to $\mathcal{S}_{ext}(\Theta)$. In particular, each FBP belongs to $\mathcal{S}_{ext}(\Theta)$ as long as Θ has infinitely many zeros.*

It might be worth comparing this result with [1, Corollary 3.4.3].

Relations (0.1) with unimodular constants $\omega = \alpha \in \mathbb{T}$

$$(0.5) \quad \frac{1 + \alpha\Theta(z)}{1 - \alpha\Theta(z)} = i\beta_\alpha + \int_{\mathbb{T}} \frac{t + z}{t - z} \sigma_\alpha(dt)$$

are well established in the theory of model spaces [3, Chapter 9] and the theory of orthogonal polynomials on the unit circle [6, Chapter 1.3]. The measures σ_α in (0.5) are known as the *Aleksandrov-Clark measures*. Our final result concerns this class of measures.

Theorem 0.5. *Let Θ be an arbitrary nonconstant inner function. Then $\sigma_\alpha \in \Sigma_{ext}(\Theta)$ for all $\alpha \in \mathbb{T}$.*

We examine the class $\Sigma_f(B)$ of measures with finite support in Section 1 and prove the main result in Section 2. In the last Section 3, given an inner function Θ , we introduce a binary operation $\mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$ (Θ -product) and show that $\sigma \notin \mathcal{S}_{ext}(\Theta)$ if and only if the Schur function $\omega = I(\sigma)$ admits a certain nontrivial factorization with respect to the Θ -product. Thereby, the “ Θ -prime functions” ω correspond to extreme points of $\Sigma(\Theta)$. The results of Theorems 0.4 and 0.5 are obtained along this line of reasoning.

1. SOME PROPERTIES OF THE CLASS $\Sigma_f(B)$

Given a FBP B of order n , we denote the divisor of its zeros by

$$\{(z_1, r_1), (z_2, r_2), \dots, (z_d, r_d)\}, \quad z_i \neq z_j, \quad i \neq j, \quad r_j \in \mathbb{N},$$

so that

$$B(z) := \prod_{k=1}^d \left(\frac{|z_k|}{z_k} \frac{z_k - z}{1 - \bar{z}_k z} \right)^{r_k}, \quad \deg B = r_1 + \dots + r_d = n.$$

The model space

$$K_B := H^2 \ominus BH^2 = \left\{ h(z) = \frac{P(z)}{\prod_{j=1}^d (1 - \bar{z}_j z)^{r_j}}, \quad \deg P \leq n-1 \right\},$$

is a finite dimensional space of all rational functions with the poles at the points $1/\bar{z}_j$ of degree at most r_j , $\dim K_B = n$. The case $z_d = 0$, i.e., $B(0) = 0$, will be of particular concern. Now

$$K_B = \left\{ h(z) = \frac{P(z)}{\prod_{j=1}^{d-1} (1 - \bar{z}_j z)^{r_j}}, \quad \deg P \leq n-1 \right\},$$

and the monomials $1, z, \dots, z^{r_{d-1}} \in K_B$. Put

$$\varphi_0(z) = 1, \quad \varphi_k(z) := \frac{1}{1 - \bar{z}_k z}, \quad k = 1, 2, \dots, d-1, \quad \varphi_d(z) = z,$$

so the standard basis in K_B is

$$(1.1) \quad \{\varphi_1, \varphi_1^2, \dots, \varphi_1^{r_1}; \dots; \varphi_{d-1}, \varphi_{d-1}^2, \dots, \varphi_{d-1}^{r_{d-1}}; \varphi_d, \dots, \varphi_d^{r_d-1}; \varphi_0\}.$$

Sometimes we reorder these functions in a unique sequence $\{e_l\}_{l=1}^n$, $e_n = 1$.

The following result is a consequence of Theorem A, but we give a simple, direct proof.

Proposition 1.1. *The support of each measure $\sigma \in \Sigma$ contains at least n points.*

Proof. If $|\text{supp } \sigma| \leq n-1$, then $\dim L_\sigma^2(\mathbb{T}) \leq n-1$, and the functions $\{e_l\}_{l=1}^n$ in (1.1) are linearly dependent in L_σ^2 , so

$$\det \|\langle e_j, e_k \rangle_\sigma\|_{j,k=1}^n = 0.$$

On the other hand, the same system is linearly independent in L_m^2 , so $\det \|\langle e_j, e_k \rangle_m\|_{j,k=1}^n \neq 0$. The contradiction completes the proof. \square

It is clear from (0.1) that a measure $\sigma \in \Sigma_f(B)$ if and only if ω is a FBP. Moreover,

$$|\operatorname{supp} \sigma| = n + \deg \omega,$$

so $|\operatorname{supp} \sigma| = n$ if and only if $\sigma = \sigma_\alpha$ is the Aleksandrov–Clark measure.

It is not hard to write $\sigma \in \Sigma_f(B)$ explicitly in terms of the corresponding parameters ω and B . Indeed, (0.1) now takes the form

$$(1.2) \quad \frac{1 + B(z)\omega(z)}{1 - B(z)\omega(z)} = i\beta + \sum_{k=1}^p \frac{t_k + z}{t_k - z} s_k,$$

and

$$(1.3) \quad \operatorname{supp} \sigma = \{t_j\}_{j=1}^p : \quad B(t_j)\omega(t_j) = 1, \quad j = 1, 2, \dots, p.$$

The weights s_j can be determined from the limit relations

$$2t_q s_q = (1 + B(t_q)\omega(t_q)) \lim_{z \rightarrow t_q} \frac{t_q - z}{1 - B(z)\omega(z)} = \frac{2}{[B\omega]'(t_q)},$$

or, in view of (1.3),

$$\frac{1}{s_q} = t_q [B\omega]'(t_q) = t_q \frac{B'(t_q)}{B(t_q)} + t_q \frac{\omega'(t_q)}{\omega(t_q)}.$$

A computation of the logarithmic derivative of a FBP is standard

$$\frac{B'(z)}{B(z)} = \sum_{k=1}^d r_k \frac{1 - |z_k|^2}{(1 - \bar{z}_k z)(z - z_k)},$$

and so

$$(1.4) \quad \frac{1}{s_q} = \sum_{k=1}^d r_k \frac{1 - |z_k|^2}{|t_q - z_k|^2} + \sum_{j=1}^m \frac{1 - |w_j|^2}{|t_q - w_j|^2},$$

where w_1, \dots, w_m are all zeros (counting multiplicity) of ω in (1.2).

Relation (1.4) provides an answer to the following “extremal mass problem”: given a point $\tau \in \mathbb{T}$, find a measure $\sigma_{\max} \in \Sigma_f(B)$ so that

$$\sigma_{\max}\{\tau\} = \max\{\sigma\{\tau\} : \sigma \in \Sigma_f(B)\}.$$

Indeed, such measure is exactly the Aleksandrov–Clark measure $\sigma = \sigma_\alpha$ with $\alpha = B^{-1}(\tau)$, $|\operatorname{supp} \sigma_{\max}| = n$, and

$$\frac{1}{\sigma_{\max}\{\tau\}} = \sum_{k=1}^d r_k \frac{1 - |z_k|^2}{|\tau - z_k|^2}.$$

Remark 1.2. As a matter of fact, the above Aleksandrov–Clark measure solves the same extremal problem within the whole class $\Sigma(B)$. Relation (1.4) holds in the form

$$\frac{1}{s_q} = \sum_{k=1}^d r_k \frac{1 - |z_k|^2}{|\tau - z_k|^2} + |\omega'(\tau)|,$$

where ω' is the angular derivative of ω (cf. [3, Section 9.2]).

Here is another simple property of measures from $\Sigma_f(B)$.

Proposition 1.3. *Let $\{t_j\}_{j=1}^p$ be an arbitrary set of distinct points on \mathbb{T} . There is a measure $\sigma \in \Sigma_f(B)$ so that*

- (1) $\{t_j\} \in \text{supp } \sigma$;
- (2) $|\text{supp } \sigma| \leq n + p - 1$.

Proof. The proof is based on the interpolation with FBP (see [5, Theorem 1]): there is a FBP ω so that $\deg \omega \leq p - 1$ and

$$\omega(t_j) = B^{-1}(t_j), \quad j = 1, \dots, p.$$

The corresponding measure σ in (1.2) is the one we need. \square

It turns out that the intersection of supports of two different measures from $\Sigma_f(B)$ can not be too large. Denote by

$$\Sigma_{n+k}(B) := \{\sigma \in \Sigma_f(B) : |\text{supp } \sigma| = n + k\}, \quad k = 0, 1, \dots$$

Lemma 1.4. *Let $\sigma_j \in \Sigma_{n+p_j}(B)$, $j = 1, 2$, and let*

$$|\text{supp } \sigma_1 \cap \text{supp } \sigma_2| \geq p_1 + p_2 + 1.$$

Then $\sigma_1 = \sigma_2$.

Proof. Let ω_j be the corresponding FBP in (1.2), $\deg \omega_j = p_j$, $j = 1, 2$. Let

$$\zeta_1, \dots, \zeta_{p_1+p_2+1} \in \text{supp } \sigma_1 \cap \text{supp } \sigma_2,$$

so, by (1.3), $\omega_1(\zeta_l) = \omega_2(\zeta_l)$, $l = 1, 2, \dots, p_1 + p_2 + 1$. Note that

$$\omega_j(z) = \gamma_j \frac{Q_j(z)}{Q_j^*(z)}, \quad j = 1, 2,$$

where γ_j are unimodular constants, Q_j are algebraic polynomials, Q_j^* are the reversed polynomials, and

$$\deg Q_j = p_j, \quad \deg Q_j^* \leq p_j, \quad j = 1, 2.$$

We see that for the polynomial

$$Q(z) = \gamma_1 Q_1(z) Q_2^*(z) - \gamma_2 Q_2(z) Q_1^*(z), \quad \deg Q \leq p_1 + p_2,$$

the relations

$$Q(\zeta_l) = 0, \quad l = 1, 2, \dots, p_1 + p_2 + 1$$

hold, so $Q = 0$, $\omega_1 = \omega_2$, and $\sigma_1 = \sigma_2$ (see Remark 0.1). \square

Corollary 1.5. *If $\sigma_j \in \Sigma_n(B)$, $j = 1, 2$, and $\text{supp } \sigma_1 \cap \text{supp } \sigma_2 \neq \emptyset$, then $\sigma_1 = \sigma_2$. If $\sigma_j \in \Sigma_{n+k}(B)$, $k = 0, 1, \dots, n - 1$, and $\text{supp } \sigma_1 = \text{supp } \sigma_2$, then $\sigma_1 = \sigma_2$.*

2. EXTREME POINTS OF $\Sigma(B)$ FOR FINITE BLASCHKE PRODUCTS

We begin with the result which provides the upper bound in (0.4). It can be viewed as a counterpart of [1, Theorem 2.3.4] for the classical moment problem.

Proposition 2.1. *Let $\sigma \in \Sigma_{ext}(B)$. Then $\sigma \in \Sigma_f(B)$ and $|\text{supp } \sigma| \leq 2n - 1$.*

Proof. Assume first that $z_d = 0$. Define an accompanying system of real valued, linearly independent functions on \mathbb{T}

$$\begin{aligned} x_{k,j}(t) &:= \operatorname{Re} \varphi_k^j(t), & y_{k,j}(t) &:= \operatorname{Im} \varphi_k^j(t), & j &= 1, \dots, r_k, & k &= 1, \dots, d-1, \\ x_{d,j}(t) &:= \operatorname{Re} t^j, & y_{d,j}(t) &:= \operatorname{Im} t^j, & j &= 1, \dots, r_d-1, & x_{d,0} &= 1. \end{aligned}$$

Reorder them in one sequence $\{v_l\}_{l=1}^{2n-1}$, and denote by E their complex, linear span

$$E := \operatorname{span}_{1 \leq l \leq 2n-1} \{v_l\}, \quad \dim E = 2n - 1.$$

Clearly,

$$\varphi_k^j = x_{k,j} + iy_{k,j} \in E, \quad \overline{\varphi_k^j} = x_{k,j} - iy_{k,j} \in E$$

(or $e_m, \overline{e_m} \in E$) for appropriate values of k, j, m , and $t^l \in E$ for $|l| \leq r_d-1$. It is a matter of a direct computation to make sure that the product $e_m \overline{e_l} \in E$, $m, l = 1, \dots, n$. For instance,

$$\begin{aligned} \varphi_p(t) \overline{\varphi_q(t)} &= \frac{1}{(1 - \bar{z}_p t)(1 - z_q \bar{t})} = \frac{\varphi_p(t) + \overline{\varphi_q(t)} - 1}{1 - \bar{z}_p z_q}, \\ \varphi_p^2(t) \overline{\varphi_q(t)} &= \frac{1}{(1 - \bar{z}_p t)^2 (1 - z_q \bar{t})} = \frac{\varphi_p^2(t) + \varphi_p(t) \overline{\varphi_q(t)} - \varphi_p(t)}{1 - \bar{z}_p z_q}, \end{aligned}$$

etc. The rest is a simple induction. We conclude, thereby, that $f\bar{g} \in E$ for each $f, g \in K_B$.

Assume next, that $|\operatorname{supp} \sigma| \geq 2n$. Then the inclusion $E \subset L_\sigma^1(\mathbb{T})$ is *proper*, so there is a nontrivial, linear functional Φ_0 on $L_\sigma^1(\mathbb{T})$, $\|\Phi_0\| \leq 1$, which vanishes on E . Equivalently, there is a function $\varphi_0 \in L_\sigma^\infty(\mathbb{T})$ such that $|\varphi_0| \leq 1$ $[\sigma]$ -almost everywhere, and

$$\int_{\mathbb{T}} x_{k,j}(t) \varphi_0(t) \sigma(dt) = \int_{\mathbb{T}} y_{k,j}(t) \varphi_0(t) \sigma(dt) = 0$$

for all appropriate values of j, k . Since the functions $x_{j,k}, y_{j,k}$ are real valued, the function φ_0 can be taken real valued as well.

Consider now two measures $\sigma_\pm(dt) := (1 \pm \varphi_0) \sigma(dt)$, $\sigma_\pm \in \mathcal{M}_+(\mathbb{T})$. By the construction, $\sigma_\pm \in \Sigma(B)$, and the representation $2\sigma = \sigma_+ + \sigma_-$ is nontrivial. Hence, σ is not an extreme point of $\Sigma(B)$, as claimed.

It remains to examine the general case when $B(0) \neq 0$. The standard trick with the change of variables (see, e.g., [6, pp.140–141]) reduces this case to the one discussed above. Given $a \in \mathbb{D}$, put

$$b_a(z) := \frac{z+a}{1+\bar{a}z}, \quad B_a(z) := B(b_a(z)), \quad \omega_a(z) := \omega(b_a(z)).$$

If we replace z with $b_a(z)$ in (0.1), we have

$$\frac{1 + B_a(z) \omega_a(z)}{1 - B_a(z) \omega_a(z)} = i\beta + \int_{\mathbb{T}} \frac{t + b_a(z)}{t - b_a(z)} \sigma(dt),$$

and since

$$\frac{t + b_a(z)}{t - b_a(z)} = i\beta_{a,t} + \frac{1 - |a|^2}{|t - a|^2} \frac{b_a(t) + z}{b_a(t) - z},$$

we come to

$$\frac{1 + B_a(z) \omega_a(z)}{1 - B_a(z) \omega_a(z)} = i\beta_a + \int_{\mathbb{T}} \frac{1 - |a|^2}{|t - a|^2} \frac{b_a(t) + z}{b_a(t) - z} \sigma(dt) = i\beta_a + \int_{\mathbb{T}} \frac{\tau + z}{\tau - z} \sigma_a(d\tau).$$

It is clear that the map $\sigma \rightarrow \sigma_a$ is a bijection of $\Sigma(B)$ onto $\Sigma(B_a)$, which is also the bijection between $\Sigma_{ext}(B)$ and $\Sigma_{ext}(B_a)$. Obviously, it is a bijection between $\Sigma_f(B)$ and $\Sigma_f(B_a)$, and in this case $|\text{supp } \sigma| = |\text{supp } \sigma_a|$. But $B_a(0) = 0$ with $a = z_d$, so the above argument applies. The proof is complete. \square

Proof of Theorem 0.2.

It remains to show that each measure $\sigma \in \Sigma_{n+k}(B)$, $k = 0, 1, \dots, n-1$ is the extreme point of $\Sigma(B)$. Indeed, let $2\sigma = \sigma_1 + \sigma_2$, then

$$\sigma_j \in \Sigma_{n+p_j}, \quad j = 1, 2, \quad 0 \leq p_1, p_2 \leq k.$$

Since $\text{supp } \sigma = \text{supp } \sigma_1 \cup \text{supp } \sigma_2$, we have

$$|\text{supp } \sigma| = |\text{supp } \sigma_1| + |\text{supp } \sigma_2| - |\text{supp } \sigma_1 \cap \text{supp } \sigma_2|,$$

or

$$|\text{supp } \sigma_1 \cap \text{supp } \sigma_2| = n + p_1 + n + p_2 - n - k = n + p_1 + p_2 - k \geq p_1 + p_2 + 1.$$

By Lemma 1.4, $\sigma_1 = \sigma_2$, so σ is the extreme point of $\Sigma(B)$, as claimed.

3. EXTREME POINTS OF $\Sigma(\Theta)$ FOR GENERIC INNER FUNCTIONS

We begin with some basics of the Nevanlinna–Pick interpolation problem in the Schur class. Given n distinct points $Z = \{z_1, \dots, z_n\}$ on the unit disk \mathbb{D} , and n complex numbers $W = \{w_1, \dots, w_n\}$, the problem is to find conditions on the interpolation data (Z, W) so that the interpolation

$$(3.1) \quad f(z_j) = w_j, \quad j = 1, 2, \dots, n$$

has at least one solution $f \in \mathcal{S}$, and to specify such conditions for (3.1) to have a unique solution.

Define a Pick matrix $\mathcal{P} = \mathcal{P}_n(Z, W)$ by

$$(3.2) \quad \mathcal{P} := \left\| \frac{1 - \bar{w}_j w_k}{1 - \bar{z}_j z_k} \right\|_{j,k=1}^n.$$

The fundamental result of G. Pick (see [4, Theorem I.2.2 and Corollary I.2.3]) looks as follows.

Theorem P. *The problem (3.1) is solvable in the class \mathcal{S} if and only if the Pick matrix (3.2) is nonnegative definite, $\mathcal{P} \geq 0$. It has a unique solution if and only if $\det \mathcal{P} = 0$.*

Our argument leans on a simple consequence of Theorem P.

Corollary 3.1. *Let $Z = \{z_1, \dots, z_n\}$ be n distinct points on \mathbb{D} , $s_0 \in \mathcal{S}$, and B be a FBP of order at most $n-1$. Assume that*

$$(3.3) \quad s_0(z_j) = B(z_j), \quad j = 1, \dots, n.$$

Then $s_0 = B$.

Proof. It is not hard to see (by the induction on the order) that for an arbitrary FBP b of order m , and a collection $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ of distinct points on \mathbb{D} with $n \geq m$, the Pick matrix (3.2) with $w_j = b(\lambda_j)$ has the rank at most m .

Put $w_j := B(z_j)$, so s_0 solves the Nevanlinna–Pick problem (3.1). Since $\deg B \leq n-1 = m$, the Pick matrix $\mathcal{P}_n(Z, W)$ has the rank at most $n-1$,

so $\det \mathcal{P} = 0$. By Theorem P, interpolation (3.3) has a unique solution in the class \mathcal{S} , so $s_0 = B$, as claimed. \square

Let us now define a binary operation on the Schur class. Given an inner function Θ and two Schur functions s_1, s_2 , put

$$(3.4) \quad (s_1 \circ s_2)_\Theta := \frac{s_0 - \Theta s_1 s_2}{1 - \Theta s_0}, \quad s_0 := \frac{s_1 + s_2}{2}.$$

$(s_1 \circ s_2)_\Theta$ will be called a Θ -product of s_1 and s_2 .

Proposition 3.2. *The Θ -product is an idempotent, binary operation on the class \mathcal{S} . $(s_1 \circ s_2)_\Theta$ is an inner function if and only if so are both s_1 and s_2 , provided that Θ is nonconstant.*

Proof. Since $1 - \Theta s_0$ is an outer function, $(s_1 \circ s_2)_\Theta$ belongs to the Smirnov class, so one has to verify that

$$|(s_1 \circ s_2)_\Theta(t)| \leq 1, \quad t \in \mathbb{T}.$$

This is a matter of direct calculation. Indeed,

$$\begin{aligned} |1 - \Theta s_0|^2 &= 1 + |s_0|^2 - \operatorname{Re}(\Theta s_1 + \Theta s_2), \\ |s_0 - \Theta s_1 s_2|^2 &= |s_0|^2 + |s_1 s_2|^2 - |s_1|^2 \operatorname{Re}(\Theta s_2) - |s_2|^2 \operatorname{Re}(\Theta s_1), \end{aligned}$$

so

$$\begin{aligned} |1 - \Theta s_0|^2 - |s_0 - \Theta s_1 s_2|^2 &= 1 - |s_1 s_2|^2 - \operatorname{Re}(\Theta s_1)(1 - |s_2|^2) - \operatorname{Re}(\Theta s_2)(1 - |s_1|^2) \\ &= 1 - |s_1 s_2|^2 - |s_1|(1 - |s_2|^2) - |s_2|(1 - |s_1|^2) \\ &\quad + (|s_1| - \operatorname{Re}(\Theta s_1))(1 - |s_2|^2) + (|s_2| - \operatorname{Re}(\Theta s_2))(1 - |s_1|^2) \\ &= (1 - |s_1 s_2|)(1 - |s_1|)(1 - |s_2|) + (|s_1| - \operatorname{Re}(\Theta s_1))(1 - |s_2|^2) \\ &\quad + (|s_2| - \operatorname{Re}(\Theta s_2))(1 - |s_1|^2) \geq 0, \end{aligned}$$

as needed.

By definition (3.4), $(s \circ s)_\Theta = s$ for each $s \in \mathcal{S}$, so the operation is idempotent.

Next, assume that $(s_1 \circ s_2)_\Theta$ is an inner function, but $|s_1| < 1$ a.e. on a set $E \subset \mathbb{T}$ of positive measure. It follows from the above calculation that

$$|s_2| = 1, \quad |s_2| - \operatorname{Re}(\Theta s_2) = 0$$

a.e. on E . Hence $\Theta s_2 = 1$ a.e. on the set of positive measure, so Θ is a unimodular constant. The contradiction completes the proof. \square

Remark 3.3. Whereas the original isometric embedding problem make no sense for constant inner functions Θ , the Θ -product is a nontrivial operation already for $\Theta = 1$. It is clear from the definition, that for $s_2 = \Theta = 1$ one has $(s_1 \circ s_2)_\Theta = 1$ for any $s_1 \in \mathcal{S}$.

Definition 3.4. A function $s \in \mathcal{S}$ is called Θ -prime if

$$s = (s_1 \circ s_2)_\Theta \Rightarrow s = s_1 = s_2.$$

The Θ -product comes in quite naturally in the context of the isometric embedding problem. Specifically, let $\sigma_1, \sigma_2 \in \Sigma(\Theta)$, and $\omega_j = I(\sigma_j)$, $j = 1, 2$,

the map I is defined in (0.2). Then obviously $\sigma = \frac{1}{2}(\sigma_1 + \sigma_2) \in \Sigma(\Theta)$. It is a matter of elementary computation based on the relation

$$\frac{1 + \Theta(z)\omega(z)}{1 - \Theta(z)\omega(z)} = \frac{1}{2} \left(\frac{1 + \Theta(z)\omega_1(z)}{1 - \Theta(z)\omega_1(z)} + \frac{1 + \Theta(z)\omega_2(z)}{1 - \Theta(z)\omega_2(z)} \right), \quad \omega = I(\sigma),$$

to check that

$$\omega = (\omega_1 \circ \omega_2)_\Theta.$$

Thereby, $\sigma \in \Sigma_{ext}(\Theta)$ if and only if $I(\sigma)$ is Θ -prime. Equivalently, ω belongs to $\mathcal{S}_{ext}(\Theta)$ if and only if ω is Θ -prime.

Proof of Theorem 0.4.

Let z_1, \dots, z_n be n distinct zeros of Θ . Given a FBP B of order at most $n - 1$, write

$$B(z) = (\omega_1 \circ \omega_2)_\Theta(z) = \frac{\omega_0(z) - \Theta(z)\omega_1(z)\omega_2(z)}{1 - \Theta(z)\omega_0(z)}$$

and so $B(z_j) = \omega_0(z_j)$, $j = 1, \dots, n$. By Corollary 3.1, $\omega_0 = B$, in particular,

$$\left| \frac{\omega_1(t) + \omega_2(t)}{2} \right| = 1, \quad \forall t \in \mathbb{T}.$$

But the latter implies $\omega_1 = \omega_2 = B$, so B is Θ -prime, as claimed.

Proof of Theorem 0.5.

Let $\omega = \gamma \in \mathbb{T}$. Write

$$\gamma = (\omega_1 \circ \omega_2)_\Theta = \frac{\omega_0 - \Theta\omega_1\omega_2}{1 - \Theta\omega_0}.$$

Solve it for ω_0

$$\omega_0 = \frac{\gamma + \Theta\omega_1\omega_2}{1 + \gamma\Theta}, \quad \omega_0 - \gamma = \Theta \frac{\omega_1\omega_2 - \gamma^2}{1 + \gamma\Theta},$$

and finally,

$$(3.5) \quad (\gamma - \omega_0)(1 + \gamma\Theta) = \Theta(\gamma^2 - \omega_1\omega_2).$$

Note that both functions $\gamma - \omega_0 = \gamma(1 - \bar{\gamma}\omega_0)$ and $1 + \gamma\Theta$ are outer, so the left hand side in (3.5) is the outer function, whereas the right hand side in (3.5) has a nontrivial inner factor. Hence,

$$\gamma^2 = \omega_1\omega_2, \quad \omega_0 = \gamma \Rightarrow \omega_1 = \omega_2 = \gamma,$$

so the constant function $\omega = \gamma$ is Θ -prime, as claimed.

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